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On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings

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Abstract

The purpose of this paper is to study the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996) 150–159; B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 73 (1967) 957–961; P.L. Lions, Approximation de points fixes de contractions, C. R. Acad. Sci. Paris, Ser. A 284 (1977), 1357–1359; S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292; Z.H. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl. 286 (2003) 351–358; R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491; H.K. Xu, M.G. Ori, An implicit iterative process for nonexpansive mappings, Numer. Funct. Anal. Optimiz. 22 (2001) 767–773; Y.Y. Zhou, S.S. Chang, Convergence of implicit iterative process

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for a finite family of asymptotically nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optimiz. 23 (2002) 911–921].

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1. Introduction and preliminaries

Throughout this paper we assume that E is a real Banach space, $F(T)$ and $D(T)$ are the set of fixed points and the domain of T, respectively.

Recall that E is said to satisfy *Opial condition*, if for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Definition 1. Let D be a closed subset of E and $T : D \to D$ be a mapping.

1. T is said to be *demi-closed* at the origin, if for each sequence $\{x_n\}$ in D, the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly implies $Tx_0 = 0$.
2. T is said to be *semi-compact*, if for any bounded sequence x_n in D such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in D$.
3. T is said to be *asymptotically nonexpansive* [3], if there exists a sequence $\{h_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant h_n \|x - y\| \quad \forall x, y \in D,\ n \geqslant 1.$$

Proposition 1.

(1) *Let K be a nonempty subset of E, $\{T_i\}_{i=1}^N : K \to K$ be N asymptotically nonexpansive mappings. Then there exists a sequence $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ such that*

$$\left\|T_i^n x - T_i^n y\right\| \leqslant h_n \|x - y\|, \quad \forall n \geqslant 1,\ x, y \in K,\ i = 1, 2, \ldots, N. \tag{1.1}$$

(2) *$\{T_1, T_2, \ldots, T_N\}$ is uniformly Lipschitzian with a Lipschitzian constant $L \geqslant 1$, i.e., there exists a constant $L \geqslant 1$ such that*

$$\left\|T_i^n x - T_i^n y\right\| \leqslant L \|x - y\|, \quad \forall n \geqslant 1,\ x, y \in K,\ i = 1, 2, \ldots, N. \tag{1.2}$$

Proof. (1) Since for each $i = 1, 2, \ldots, N$, $T_i : K \to K$ is an asymptotically nonexpansive mapping, there exists a sequence $\{h_n^{(i)}\} \subset [1, \infty)$, with $h_n^{(i)} \to 1$ $(n \to \infty)$ such that

$$\left\|T_i^n x - T_i^n y\right\| \leqslant h_n^{(i)} \|x - y\|, \quad \forall n \geqslant 1. \tag{1.3}$$

Letting

$$h_n = \max\{h_n^{(1)}, h_n^{(2)}, \dots, k_n^{(N)}\}, \tag{1.4}$$

then we have that $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ $(n \to \infty)$ and

$$\|T_i^n x - T_i^n y\| \leqslant h_n^{(i)} \|x - y\| \leqslant h_n \|x - y\|, \quad \forall n \geqslant 1,$$

for all $x, y \in K$ and for each $i = 1, 2, \dots, N$.

(2) Taking $L = \sup_{n \geqslant 1} h_n$, then the conclusion (2) can be obtained from the conclusion (1) immediately.

Definition 2. Let K be a nonempty closed convex subset of E satisfying $K + K \subset K$, $x_0 \in K$ be any given point and $\{T_1, T_1, \dots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings. Let $\{\alpha_n\}$ be a sequence in [0, 1] and $\{u_n\}$ be a bounded sequence in K. Then the sequence $\{x_n\} \subset K$ defined by

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 + u_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 + u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N + u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1} + u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N} + u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1} + u_{2N+1}, \\
&\vdots
\end{aligned}$$

is called the implicit iterative sequence with errors for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$.

Since for each $n \geqslant 1$, it can be written as $n = (k - 1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\}$, $k = k(n) \geqslant 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$. Hence we can write the above table in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n + u_n, \quad \forall n \geqslant 1. \tag{1.5}$$

Especially, if $T_1, T_2, \dots, T_N : K \to K$ are N asymptotically nonexpansive mappings, $\{\alpha_n\}$ is a sequence in [0, 1] and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad \forall n \geqslant 1, \tag{1.6}$$

is called the implicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$.

Recently concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, for example, Bauschke [1], Goebel and Kirk [3], Gornicki [4], Halpern [5], Lions [6], Reich [7], Schu [8], Sun [9], Tan and Xu [10], Wittmann [12], Xu and Ori [13], Zhou and Chang [14]).

The purpose of this paper is to study the weak and strong convergence of implicit iteration sequences $\{x_n\}$ defined by (1.5) and (1.6) to a common fixed point for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces.

The following theorems are the main results of this paper.

Theorem 1. *Let E be a real uniformly convex Banach space satisfying Opial condition, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \dots, T_N\}$). Let $\{u_n\}$ be a bounded sequence in K, $\{\alpha_n\}$ be a sequences in* $[0, 1]$ *and $\{h_n\}$ be the sequence defined by* (1.1) *and $L = \sup_{n\geqslant 1} h_n \geqslant 1$ satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$;
(ii) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$;
(iii) *there exist constants $\tau_1, \tau_2 \in (0, 1)$ such that*

$$\tau_1 \leqslant (1 - \alpha_n) \leqslant \tau_2, \quad \forall n \geqslant 1.$$

Then the implicit iterative sequence $\{x_n\}$ defined by (1.5) *converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.*

Theorem 2. *Let E be a real uniformly convex Banach space satisfying Opial condition, K be a nonempty closed convex subset of E, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and $\{\alpha_n\}$ be a sequences in* $[0, 1]$ *and $\{h_n\}$ be the sequence defined by* (1.1) *and $L = \sup_{n\geqslant 1} h_n \geqslant 1$ satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$;
(ii) *there exist constants $\tau_1, \tau_2 \in (0, 1)$ such that*

$$\tau_1 < 1 - \alpha_n < \tau_2.$$

Then the implicit iterative sequence $\{x_n\}$ defined by (1.6) *converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.*

Theorem 3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and at least there exists an $T_l, 1 \leqslant l \leqslant N$, which is semi-compact* (*without loss of generality, we can assume that T_1 is semi-compact*)*. Let $\{u_n\}$ be a bounded sequence in K, $\{\alpha_n\}$ be a sequences in* $[0, 1]$ *and $\{h_n\}$ be the sequence defined by* (1.1) *and $L = \sup_{n\geqslant 1} h_n \geqslant 1$ satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$;

(ii) $\sum_{n=1}^{\infty}(h_n - 1) < \infty$;
(iii) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that*

$$\tau_1 \leqslant (1-\alpha_n) \leqslant \tau_2, \quad \forall n \geqslant 1.$$

Then the implicit iterative sequence $\{x_n\}$ *defined by* (1.5) *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *in* K.

Theorem 4. *Let* E *be a real uniformly convex Banach space,* K *be a nonempty closed convex subset of* E, $\{T_1, T_2, \ldots, T_N\} : K \to K$ *be* N *asymptotically nonexpansive mappings with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and at least there exists an* T_l, $1 \leqslant l \leqslant N$ *is semi-compact. Let* $\{\alpha_n\}$ *be a sequences in* $[0,1]$, $\{h_n\}$ *be the sequence defined by* (1.1) *and* $L = \sup_{n \geqslant 1} h_n \geqslant 1$ *satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty}(h_n - 1) < \infty$;
(ii) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that*

$$\tau_1 < 1 - \alpha_n < \tau_2.$$

Then the implicit iterative sequence $\{x_n\}$ *defined by* (1.6) *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *in* K.

Theorem 5. *Let* E *be a real uniformly convex Banach space,* K *be a nonempty closed convex subset of* E *with* $K + K \subset K$, $\{T_1, T_2, \ldots, T_N\} : K \to K$ *be* N *nonexpansive mappings with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{u_n\}$ *be a bounded sequence in* K, $\{\alpha_n\}$ *be a sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$;
(ii) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that*

$$\tau_1 \leqslant (1-\alpha_n) \leqslant \tau_2, \quad \forall n \geqslant 1.$$

(1) *if there exists at least an* T_l, $1 \leqslant l \leqslant N$, *which is semi-compact, then the implicit iterative sequence* $\{x_n\}$ *defined by* (1.5) *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *in* K;
(2) *if* E *is semi-closed, then the implicit iterative sequence* $\{x_n\}$ *defined by* (1.5) *converges weakly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *in* K.

In order to prove the main results of this paper, we need the following lemmas.

Lemma 1 [2,4]. *Let* E *be a uniformly convex Banach space,* K *be a nonempty closed convex subset of* E *and* $T : K \to K$ *be an asymptotically nonexpansive mapping. Then* $I - T$ *is semi-closed at zero, i.e., for each sequence* $\{x_n\}$ *in* K, *if* $\{x_n\}$ *converges weakly to* $q \in K$ *and* $\{(I-T)x_n\}$ *converges strongly to* 0, *then* $(I-T)q = 0$.

Lemma 2 [11]. *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leqslant (1+b_n)a_n + c_n, \quad \forall n \geqslant n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$.

Then

(1) *the limit* $\lim_{n\to\infty} a_n$ *exists*;
(2) *if, in addition, there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ (as $n \to \infty$).*

Lemma 3 [8]. *Let E be a uniformly convex Banach space, b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in E. Then the conditions*:

$$\begin{cases} \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \\ \limsup_{n\to\infty} \|x_n\| \leqslant d, \\ \limsup_{n\to\infty} \|y_n\| \leqslant d \end{cases}$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, *where $d \geqslant 0$ is some constant.*

2. Proof of theorems

We are now in a position to prove our main results in this paper.

Proof of Theorem 1. Since $F = \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (1.5) and Proposition 1 that

$$\begin{aligned} \|x_n - p\| &= \left\|\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + u_n - p\right\| \\ &\leqslant \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\left\|T_{i(n)}^{k(n)} x_n - p\right\| + \|u_n\| \\ &\leqslant \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)h_{k(n)}\|x_n - p\| + \|u_n\|. \end{aligned}$$

Letting $\mu_n = h_{k(n)} - 1$, $\forall n \geqslant 1$, by condition (i) we have

$$\sum_{i=1}^{\infty} \mu_n < \infty. \tag{2.1}$$

Therefore we have

$$\begin{aligned} \|x_n - p\| &\leqslant \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)(1+\mu_n)\|x_n - p\| + \|u_n\| \\ &\leqslant \alpha_n \|x_{n-1} - p\| + (1-\alpha_n+\mu_n)\|x_n - p\| + \|u_n\|. \end{aligned}$$

Simplifying we have

$$\|x_n - p\| \leqslant \|x_{n-1} - p\| + \frac{\mu_n}{\alpha_n}\|x_n - p\| + \frac{\|u_n\|}{\alpha_n}. \tag{2.2}$$

By condition (iii), $1-\tau_2 \leqslant \alpha_n$, hence from (2.2) we have

$$\|x_n - p\| \leqslant \|x_{n-1} - p\| + \frac{\mu_n}{1-\tau_2}\|x_n - p\| + \frac{\|u_n\|}{1-\tau_2}.$$

Simplifying we have

$$\begin{aligned}\|x_n - p\| &\leqslant \frac{1-\tau_2}{1-\tau_2-\mu_n}\|x_{n-1} - p\| + \frac{\|u_n\|}{(1-\tau_2-\mu_n)(1-\tau_2)} \\ &= \left(1 + \frac{\mu_n}{1-\tau_2-\mu_n}\right)\|x_{n-1} - p\| + \frac{\|u_n\|}{(1-\tau_2-\mu_n)(1-\tau_2)}. \qquad (2.3)\end{aligned}$$

By virtue of (2.1), $\mu_n \to 0$, therefore there exists a positive integer n_0 such that $\mu_n \leqslant \frac{1-\tau_2}{2}$, $\forall n \geqslant n_0$. It follows from (2.3) that

$$\|x_n - p\| \leqslant \left(1 + \frac{2\mu_n}{1-\tau_2}\right)\|x_{n-1} - p\| + \frac{2\|u_n\|}{(1-\tau_2)(1-\tau_2)}, \quad \forall n \geqslant n_0. \qquad (2.4)$$

Taking $a_{n+1} = \|x_n - p\|$, $b_n = \frac{2\mu_n}{1-\tau_2}$, $c_n = \frac{2\|u_n\|}{(1-\tau_2)(1-\tau_2)}$ in Lemma 2 and by using conditions (i) and (2.1), it is easy to see that

$$\sum_{n=1}^{\infty} b_n < \infty; \qquad \sum_{n=1}^{\infty} c_n < \infty.$$

It follows from Lemma 2 that the $\lim_{n\to\infty} \|x_n - p\|$ exists. Without loss of generality we can assume that

$$\lim_{n\to\infty} \|x_n - p\| = d, \qquad (2.5)$$

where $d \geqslant 0$ is some number. Since $\{\|x_n - p\|\}$ is a convergent sequence and so $\{x_n\}$ is a bounded sequence in K. Again since

$$\|x_n - p\| = \left\|\alpha_n[x_{n-1} - p + u_n] + (1-\alpha_n)\left[T_{i(n)}^{k(n)} x_n - p + u_n\right]\right\|.$$

By condition (i) and (2.5) we have that

$$\limsup_{n\to\infty} \|x_{n-1} - p + u_n\| \leqslant \limsup_{n\to\infty} \|x_{n-1} - p\| + \limsup_{n\to\infty} \|u_n\| \leqslant d; \qquad (2.6)$$

and that

$$\limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} x_n - p + u_n\right\| \leqslant \limsup_{n\to\infty} h_{k(n)}\|x_n - p\| + \limsup_{n\to\infty} \|u_n\| \leqslant d. \qquad (2.7)$$

Therefore from (2.5)–(2.7) and Lemma 3 we have that

$$\lim_{n\to\infty} \left\|T_{i(n)}^{k(n)} x_n - x_{n-1}\right\| = 0. \qquad (2.8)$$

Moreover, since

$$\begin{aligned}\|x_n - x_{n-1}\| &= \left\|(1-\alpha_n)T_{i(n)}^{k(n)} x_n - (1-\alpha_n)x_{n-1} + u_n\right\| \\ &\leqslant (1-\alpha_n)\left\|T_{i(n)}^{k(n)} x_n - x_{n-1}\right\| + \|u_n\|,\end{aligned}$$

it follows from (2.8) and condition (i) that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.9}$$

From (2.8) and (2.9) we have

$$\lim_{n\to\infty}\big\|x_n - T_{i(n)}^{k(n)}x_n\big\| \leqslant \lim_{n\to\infty}\big\{\|x_n - x_{n-1}\| + \big\|x_{n-1} - T_{i(n)}^{k(n)}x_n\big\|\big\} = 0, \tag{2.10}$$

and

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0, \quad \forall j = 1, 2, \dots, N. \tag{2.11}$$

Since for any positive integer $n > N$, it can be written as $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, \dots, N\}$. Letting $\sigma_n = \|T_{i(n)}^{k(n)}x_n - x_{n-1}\|$, then from (2.8), we have $\sigma_n \to 0$ and

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leqslant \big\|x_{n-1} - T_{i(n)}^{k(n)}x_n\big\| + \big\|T_{i(n)}^{k(n)}x_n - T_n x_n\big\| \\
&= \sigma_n + \big\|T_{i(n)}^{k(n)}x_n - T_{i(n)}x_n\big\| \leqslant \sigma_n + L\big\|T_{i(n)}^{k(n)-1}x_n - x_n\big\| \\
&\leqslant \sigma_n + L\big\{\big\|T_{i(n)}^{k(n)-1}x_n - T_{n-N}^{k(n)-1}x_{n-N}\big\| \\
&\quad + \big\|T_{n-N}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\big\| + \|x_{(n-N)-1} - x_n\|\big\}.
\end{aligned} \tag{2.12}$$

Since for each $n > N$, $n = (n-N)(\mathrm{mod}\ N)$, again since $n = (k(n)-1)N + i(n)$, hence $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, i.e.,

$$k(n-N) = k(n) - 1 \quad \text{and} \quad i(n-N) = i(n).$$

Therefore we have

$$\big\|T_{i(n)}^{k(n)-1}x_n - T_{n-N}^{k(n)-1}x_{n-N}\big\| = \big\|T_{i(n)}^{k(n)-1}x_n - T_{i(n)}^{k(n)-1}x_{n-N}\big\| \leqslant L\|x_n - x_{n-N}\| \tag{2.13}$$

and

$$\big\|T_{n-N}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\big\| = \big\|T_{i(n-N)}^{k(n-N)}x_{(n-N)} - x_{(n-N)-1}\big\| = \sigma_{n-N}. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12) and simplifying we have

$$\|x_{n-1} - T_n x_n\| \leqslant \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_{(n-N)-1} - x_n\|.$$

By (2.8) and (2.11) we know that

$$\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0. \tag{2.15}$$

It follows from (2.9) and (2.15) that

$$\lim_{n\to\infty}\|x_n - T_n x_n\| \leqslant \lim_{n\to\infty}\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\| = 0. \tag{2.16}$$

Consequently, for any $j = 1, 2, \dots, N$ from (2.11) and (2.16) we have

$$\begin{aligned}
\|x_n - T_{n+j}x_n\| &\leqslant \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\
&\leqslant (1+L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| \to 0,
\end{aligned}$$

$$\text{as } n \to \infty. \tag{2.17}$$

This implies that the sequence

$$\bigcup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\}_{n=1}^{\infty} \to 0 \quad \text{as } n \to \infty.$$

Since for each $l = 1, 2, \ldots, N$, $\{\|x_n - T_l x_n\|\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\}_{n=1}^{\infty}$, therefore we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.18}$$

Since E is uniformly convex, every bounded subset of E is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence from (2.18) we have

$$\lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.19}$$

By Lemma 1, we have that $(I - T_l)q = 0$, i.e., $q \in F(T_l)$. By the arbitrariness of $l \in \{1, 2, \ldots, N\}$, we know that $q \in F = \bigcap_{l=1}^{N} F(T_l)$.

Next we prove that $\{x_n\}$ converges weakly to q. Suppose the contrary, then there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q \neq q_1$. Then by the same method as given above, we can also prove that $q_1 \in F = \bigcap_{l=1}^{N} F(T_l)$. Taking $p = q$ and $p = q_1$ and by using the same method as given in the proof of (2.5) we can prove that the following two limits exist and

$$\lim_{n\to\infty} \|x_n - q\| = d_1, \qquad \lim_{n\to\infty} \|x_n - q_1\| = d_2,$$

where d_1, d_2 are two nonnegative numbers. By virtue of the Opial condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k\to\infty} \|x_{n_k} - q\| < \limsup_{n_k\to\infty} \|x_{n_k} - q_1\| = \limsup_{n_j\to\infty} \|x_{n_j} - q_1\| \\ &< \limsup_{n_j\to\infty} \|x_{n_j} - q\| = d_1. \end{aligned}$$

This is a contradiction. Hence $q = q_1$. This implies that $\{x_n\}$ converges weakly to q.

The proof of Theorem 1 is completed. □

Proof of Theorem 2. Taking $\gamma_n = 0$, $\forall n \geqslant 1$, and $\beta_n = 1 - \alpha_n$, $\forall n \geqslant 1$, in Theorem 1, then the conclusion of Theorem 2 can be obtained from Theorem 1 immediately. □

Proof of Theorem 3. For any given $p \in F = \bigcap_{i=1}^{N} F(T_i)$, by the same method as given in proving (2.5) and (2.18), we can prove that

$$\lim_{n\to\infty} \|x_n - p\| = d, \tag{2.20}$$

where $d \geqslant 0$ is some nonnegative number, and

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.21}$$

Especially, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.22}$$

By the assumption of Theorem 3, T_1 is semi-compact, therefore it follows from (2.22) that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x_* \in K$. Hence from (2.21) we have that

$$\|x_* - T_l x_*\| = \lim_{n_i \to \infty} \|x_{n_i} - T_l x_{n_i}\| = 0, \quad \forall l = 1, 2, \ldots, N.$$

This implies that

$$x_* \in F = \bigcap_{i=1}^{N} F(T_i).$$

By the arbitrariness of $p \in F = \bigcap_{i=1}^{N} F(T_i)$, in (2.20) taking $p = x_*$, similarly we can also prove that

$$\lim_{n \to \infty} \|x_n - x_*\| = d_1,$$

where $d_1 \geqslant 0$ is some nonnegative number. From $x_{n_i} \to x_*$ we know that $d_1 = 0$, i.e., $x_n \to x_*$.

This completes the proof of Theorem 3. □

Proof of Theorem 4. Taking $\gamma_n = 0$ and $\beta_n = 1 - \alpha_n$, $\forall n \geqslant 1$ in Theorem 3, the conclusion of Theorem 4 can be obtained from Theorem 3 immediately. □

Proof of Theorem 5. Since each nonexpansive mapping from K into K is an asymptotically nonexpansive mapping from $K \to K$ with $h_n = 1$, $\forall n \geqslant 1$ and $L = 1$. Therefore all conditions in Theorems 1 and 3 are satisfied. The conclusions of Theorem 5 can be obtained from Theorems 1 and 3 immediately.

This completes the proof of Theorem 5. □

Remark. (1) Theorems 2 and 4 give an affirmative answer to the following open question raised by Xu and Ori [13]: "It is unclear what assumptions on the mappings $\{T_1, T_2, \ldots, T_N\}$ and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$ defined by (1.6)."

(2) Theorems 1 and 3 improve and generalize Theorem 3.3 of Sun [9] to the case of implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings and the boundedness condition of the set K in this theorem is deleted.

(3) Theorems 1 and 3 also generalize and improve the corresponding results in Zhou and Chang [14] and the key condition (v) in [14, Theorem 1]: there exists a constant $L > 0$ such that for any $i, j \in \{T_1, T_2, \ldots, N\}$, $i \neq j$,

$$\left\| T_i^n x - T_j^n \right\| \leqslant L\|x - y\|, \quad \forall n \geqslant 1, \ \forall x, y \in K,$$

is deleted.

(4) Theorems 1–5 also generalize and improve the corresponding results of [1,3–8, 10,12].

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